

ALMOST ISOLATED SPECTRAL PARTS AND INVARIANT SUBSPACES⁽¹⁾

BY

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ABSTRACT. Let T be an operator with spectrum $\sigma(T)$ on a Hilbert space. A compact subset E of $\sigma(T)$ is called a disconnected part of $\sigma(T)$ if, for some bounded open set A , E is the closure of $\sigma(T) \cap A$ and $\sigma(T) - E$ is the union of the isolated parts of $\sigma(T)$ lying completely outside the closure of A . The set E is called an almost isolated part of $\sigma(T)$ if, in addition, the set A can be chosen so as to have a rectifiable boundary ∂A on which the subset $\sigma(T) \cap \partial A$ has arc length measure 0. The following results are obtained. If T is subnormal and if E is a disconnected part of $\sigma(T)$ then there exists a reducing subspace \mathfrak{M} of T for which $\sigma(T|_{\mathfrak{M}}) = E$. If T^* is hyponormal and if E is an almost isolated part of $\sigma(T)$ then there exists an invariant subspace \mathfrak{M} of T for which $\sigma(T|_{\mathfrak{M}}) = E$. An example is given showing that if T is arbitrary then an almost isolated part of $\sigma(T)$ need not be the spectrum of the restriction of T to any invariant subspace.

1. Introduction. Let T denote a bounded operator on a Hilbert space \mathfrak{H} with spectrum $\sigma(T)$ and resolvent set $\rho(T)$. A bounded open set D of the complex plane is called an admissible domain of T (cf. Riesz and Sz.-Nagy [14, p. 418]) if its boundary ∂D consists of a finite number of rectifiable closed curves lying in $\rho(T)$ and positively oriented with respect to D . A compact subset σ of $\sigma(T)$ is called an isolated part of $\sigma(T)$ if its distance from $\sigma(T) - \sigma$ is positive, in which case there exists some admissible domain D for which

$$(1.1) \quad \sigma = \sigma(T) \cap D.$$

Moreover, if σ is nonempty then the Riesz integral

$$(1.2) \quad P_{\sigma} = -(2\pi i)^{-1} \int_{\partial D} (T - z)^{-1} dz$$

defines a projection ($P_{\sigma}^2 = P_{\sigma}$) for which the space $\mathfrak{M}_{\sigma} = P_{\sigma}(\mathfrak{H})$ is invariant under T and $\sigma(T|_{\mathfrak{M}_{\sigma}}) = \sigma$. In fact, \mathfrak{M}_{σ} is a hyperinvariant subspace of T , that is, \mathfrak{M}_{σ} is

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invariant under any bounded operator which commutes with T .

A compact set E of $\sigma(T)$ will be called a *disconnected part* of $\sigma(T)$ if there exists a bounded open set A such that

$$(1.3) \quad E = (\sigma(T) \cap A)^- \quad \text{and} \quad \sigma(T) - E = \bigcup \{ \sigma : \sigma \subset \mathbb{C} - A^-, \}$$

where the union is taken over all isolated parts of $\sigma(T)$ which lie completely outside the closure, A^- , of A . Such a set E will be called an *almost isolated part* (a.i.p.) of $\sigma(T)$ if, in addition, the set A can be chosen so that ∂A consists of a finite number of rectifiable closed curves oriented positively with respect to A and satisfying

$$(1.4) \quad \text{meas}_1(\sigma(T) \cap \partial A) = 0,$$

where the measure denotes arc length on ∂A . Clearly, an isolated part E of $\sigma(T)$ is also an a.i.p. of $\sigma(T)$, while the converse holds if and only if the set A satisfying (1.3) and (1.4) can be chosen so that also $\sigma(T) \cap \partial A = \emptyset$. Note that if, for instance,

$$(1.5) \quad \sigma(T) = \{z: |z + 1| \leq 1\} \cup \{1, 1/2, 1/3, \dots\},$$

then $E = \{z: |z + 1| \leq 1\}$ is an a.i.p. of $\sigma(T)$ but not an isolated part. Similarly, $[-1, 0]$ is an a.i.p. of $\sigma(T) = [-1, 0] \cup \{1, 1/2, 1/3, \dots\}$; on the other hand, $\{0\}$ is not an a.i.p. of $\sigma(T) = \{0\} \cup \{1, 1/2, 1/3, \dots\}$.

Obviously, a set E is a disconnected part or an almost isolated part of $\sigma(T)$ if and only if the set $\{\bar{z}: z \in E\}$ bears the corresponding relation to $\sigma(T^*)$.

In general, an a.i.p. of $\sigma(T)$ need not be the spectrum of any restriction of T to an invariant subspace. In fact, it will be shown (§5) that there exists an operator T for which (1.5) holds and such that if \mathfrak{M} is any nontrivial invariant subspace of T then

$$(1.6) \quad \sigma(T|_{\mathfrak{M}}) \cap \{1, 1/2, 1/3, \dots\} \neq \emptyset.$$

We recall some definitions and properties. An operator T is said to be hyponormal if $T^*T - TT^* \geq 0$, in which case it is known (Stampfli [16]) that

$$(1.7) \quad \|(T - z)x\| \geq \|(T^* - \bar{z})x\| \geq [\text{dist}(z, \sigma(T))] \|x\|.$$

The restriction of a hyponormal operator to an invariant subspace is also hyponormal; see Berberian [1, p. 161]. An operator T is said to be subnormal if it has a normal extension on a larger Hilbert space. The subnormal operators form a proper subset of the hyponormal operators; see, e.g., Halmos [6, p. 105].

A hyponormal operator is called completely hyponormal if it has no normal part, that is, if there does not exist a (nontrivial) reducing subspace of T on which it is normal. Similarly, one can define a completely subnormal operator. It is

known that if T is completely hyponormal and if D is any open disk for which $\sigma(T) \cap D$ is nonempty then $\sigma(T) \cap D$ must have positive planar measure (Putnam [9]). Conversely, if α is any compact set with the property that for any open disk D the set $\alpha \cap D$ has positive planar measure whenever it is nonempty, then α is the spectrum of some completely hyponormal operator; see Putnam [12]. In Carey and Pincus [4] it is shown that, in fact, the operator T can be chosen so that $T^*T - TT^*$ even has rank 1. For necessary and sufficient conditions that a compact set be the spectrum of a completely subnormal operator, see Clancey and Putnam [2].

For a summary of properties of certain "almost normal" operators, see Putnam [11].

Let T be normal and let α denote any (nonempty) compact subset of $\sigma(T)$ of the form $\alpha = (\sigma(T) \cap A)^-$ where A is an open set. Then it is easy to see that there exists an invariant subspace \mathfrak{M} of T for which $\sigma(T|_{\mathfrak{M}}) = \alpha$. In fact, if E_z is the spectral family of T then $\mathfrak{M} = E(\alpha)\mathfrak{H}$ has the required property. That no such property can hold for hyponormal or even subnormal operators is clear. In fact, one need only consider a completely subnormal isometry, say, the unilateral shift. Its spectrum is the closed unit disk and if \mathfrak{M} is any (non-trivial) invariant subspace then the restriction of this shift to \mathfrak{M} is also a completely subnormal isometry with the same spectrum. It is interesting though to note that if T is an arbitrary contraction ($\|T\| \leq 1$) on a Hilbert space \mathfrak{H} , then there exists an isometry V on a larger Hilbert space such that T is the restriction of V^* to \mathfrak{H} ; see Sz.-Nagy and Foiaş [19, p. 11]. In particular, any compact subset of the closed unit disk is the spectrum of some coisometry restricted to an invariant subspace. See also Sz.-Nagy and Foiaş [19, pp. 66ff] and Douglas, Muhly and Pearcy [5].

2. Subnormal operators. Let T be subnormal and suppose that σ_1 and $\sigma_2 = \sigma(T) - \sigma_1$ are nonempty isolated parts of $\sigma(T)$. As already noted, there exist invariant subspaces $\mathfrak{M}_k = P_k(\mathfrak{H})$ ($P_k = P_{\sigma_k}$ of (1.2)) for which $\sigma(T|_{\mathfrak{M}_k}) = \sigma_k$ ($k = 1$ and 2). Since T is subnormal, much more can be said however. In fact, P_k is selfadjoint, \mathfrak{M}_k reduces T , $T|_{\mathfrak{M}_k}$ is subnormal and

$$(2.1) \quad T = T|_{\mathfrak{M}_1} \oplus T|_{\mathfrak{M}_2};$$

see Williams [20, pp. 97–98]. These facts can be used to establish

THEOREM 1. *Let T be subnormal and let E ($\neq \emptyset, \sigma(T)$) be a disconnected part of $\sigma(T)$. Then there exist subspaces \mathfrak{M}_k ($k = 1, 2$), which are hyperinvariant and reducing for T , and for which $T|_{\mathfrak{M}_k}$ is subnormal, $\sigma(T|_{\mathfrak{M}_1}) = E$ and $\sigma(T|_{\mathfrak{M}_2}) = (\sigma(T) - E)^-$, and (2.1) holds.*

PROOF. According to the definition of a disconnected part of $\sigma(T)$, there exists a bounded open set A satisfying condition (1.3). Clearly, $\sigma(T) - E = \bigcup \sigma_k$ where $\sigma_1, \sigma_2, \dots$ is a (possibly finite) sequence of pairwise disjoint isolated parts of $\sigma(T)$. Then the operators $P_k = P_{\sigma_k}$ are selfadjoint and pairwise orthogonal, the spaces $\mathfrak{M}_k = P_k(\mathfrak{H})$ are hyperinvariant, reducing subspaces of T , and the operators $T|_{\mathfrak{M}_k}$ are subnormal. If $Q = \sum \bigoplus P_k$ and $R = I - Q$, then Q and R are orthogonal (selfadjoint) projections, $\mathfrak{M}_1 = R(\mathfrak{H})$ and $\mathfrak{M}_2 = Q(\mathfrak{H})$ are hyperinvariant, reducing subspaces of T , and (2.1) holds, where both summands are subnormal. Since $T|_{\mathfrak{M}_2} = \sum \bigoplus (T|_{P_k(\mathfrak{H})})$ and $\sigma(T|_{P_k(\mathfrak{H})}) = \sigma_k$, then $(\sigma(T) - E)^- \subset \sigma(T|_{\mathfrak{M}_2})$. The reverse inclusion, hence equality, follows from (1.7). That $\sigma(T|_{\mathfrak{M}_1}) \supset E$ is clear from (1.3) and (2.1). If $R_n = I - (P_1 + \dots + P_n)$ then (cf. the beginning of §1) $\sigma(T|R_n(\mathfrak{H})) = \sigma(T) - (\sigma_1 \cup \dots \cup \sigma_n)$. Since $R_n \rightarrow R$ (strongly), another appeal to (1.7) shows that $\sigma(T|_{\mathfrak{M}_1}) = E$. This completes the proof of Theorem 1.

3. **Hyponormal operators.** It is interesting to contrast the above situation for subnormal operators with the corresponding one for hyponormal operators. As was indicated above, if T is subnormal and if $\sigma(T)$ is the union of two non-empty disjoint compact sets (hence, each is an isolated part of $\sigma(T)$), then T is reducible. In particular, if $T^*T - TT^*$ has rank 1, then T has a normal part. On the other hand, it is possible that T is completely hyponormal, $\sigma(T)$ is the union of disjoint, nonempty isolated parts σ_1 and σ_2 , but T is irreducible. A simple example is furnished by the singular integral operator $T = H + iJ$ on $\mathfrak{H} = L^2(\alpha)$, where $\alpha = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and where H and J are the selfadjoint operators

$$(Hx)(t) = tx(t), \quad (Jx)(t) = -(i\pi)^{-1} \int_{\alpha} (s - t)^{-1} x(s) ds.$$

(Here, the integral operator is interpreted as a Cauchy principal value integral.) It is easy to verify that T is hyponormal and that $T^*T - TT^*$ has rank 1. In addition, it turns out that $\sigma(T)$ consists of the two isolated parts $\sigma_1 = [0, \frac{1}{4}] \times [-1, 1]$ and $\sigma_2 = [\frac{3}{4}, 1] \times [-1, 1]$, and that, further, T is irreducible; see Clancey and Putnam [3, p. 452].

Let T be hyponormal and suppose that σ is a nonempty isolated part of $\sigma(T)$. Then, as noted above, there exists an invariant subspace \mathfrak{M} of T for which $(T|_{\mathfrak{M}}$ is hyponormal and) $\sigma(T|_{\mathfrak{M}}) = \sigma$. Whether a similar assertion concerning the spectrum holds if σ is replaced by an almost isolated part E will, in general, remain open. However, the following will be proved.

THEOREM 2. *Let T^* be hyponormal, and let E be an almost isolated part of $\sigma(T)$ contained in an associated bounded open set A considered in §1 and satisfying (1.3) and (1.4). Then there exists a hyperinvariant subspace \mathfrak{M} of T for*

which

$$(3.1) \quad \sigma(T| \mathfrak{M}) = E.$$

REMARKS. We do not know whether the assertion of Theorem 2 is still valid if, for instance, either the hypothesis (1.4) is omitted (so that E is assumed only to be a disconnected part of $\sigma(T)$) or if the hypothesis that T^* be hyponormal is replaced by the hypothesis that T be hyponormal. Of course, if T is hyponormal and if E is an almost isolated part of $\sigma(T)$, then Theorem 2 asserts the existence of a hyperinvariant subspace \mathfrak{N} of T^* for which $\sigma(T^*| \mathfrak{N}) = \{\bar{z} : z \in E\}$. It is noteworthy also that if T is hyponormal and if E is a disconnected part of $\sigma(T)$ for which $E \neq \sigma(T)$, then there exists a hyperinvariant subspace \mathfrak{M} of T for which $\sigma(T| \mathfrak{M}) = (\sigma(T) - E)^-$. This can be deduced from a result of Stampfli [18] (cf. Theorem 2 there). Of course, if T is subnormal, even more can be claimed (cf. Theorem 1 above).

A related problem concerns the existence of invariant subspaces of operators T for which $\sigma(T)$ is connected but is such that $\sigma(T) = E \cup F$, where E and F are compact and $E \cap F$ has linear (arc length) measure 0 as a subset of a rectifiable curve. In this connection, see Lautzenheiser [7] and Putnam [13], where the operators considered are subnormal, and Stampfli [17], where the operators satisfy certain resolvent growth conditions, and certain restrictions are also imposed on the degree of contact between E and F . In [17], one can also find numerous references.

Before beginning the proof of Theorem 2, it will be convenient to recall some results obtained in Putnam [10]. Let T^* be hyponormal so that

$$(3.2) \quad TT^* - T^*T = D \geq 0.$$

If D has the spectral family G_t and if $x = G(s, \infty)x$, where $s > 0$, then for each vector y in \mathfrak{H} , the function $F(z) = ((T - z)^{-1}x, y)$ is, in addition to being analytic for $z \in \rho(T)$, also continuous and bounded in $\mathbb{C} - \Pi$, where Π is the set of complex z for which either z is in the point spectrum of T or \bar{z} is in the point spectrum of T^* . (The set P occurring in Putnam [10, p. 165], also in [11, p. 622], should be defined as Π above.) By an adaptation of that argument we can obtain other vectors x for which $(T - z)^{-1}x$ is (weakly) continuous and bounded, as above, at least for certain parts of the plane including portions of $\sigma(T)$.

To see this, let T have the rectangular representation $T = H + iJ$, where H has the spectral resolution $H = \int \lambda dE_\lambda$. Then, if $z = t + is$, one has by (3.2),

$$(3.3) \quad (T - z)(T - z)^* = (H - t)^2 + (J - s)^2 + \frac{1}{2}D \geq (H - t)^2.$$

An argument like that in [10, p. 166], shows that if α is any Borel set of the

real line and if z is not in Π and t is not in α^- then $\|(T - z)^{-1}x\|^2 \leq \int_{\alpha} (\lambda - t)^2 d\|E_{\lambda}x\|^2$. (Incidentally, if t is not in the point spectrum of H , this last inequality holds for arbitrary Borel sets α provided x is in the domain of $(H - t)^{-1}$. This situation surely obtains if T is completely hyponormal, in which case H is even absolutely continuous; cf. [8, p. 43].) Hence,

$$(3.4) \quad \|(T - z)^{-1}E(\alpha)x\| \leq [\text{dist}(\text{Re}(z), \alpha)]^{-1}\|E(\alpha)x\|, \quad z \notin \Pi \text{ and } \text{Re}(z) \notin \alpha^-.$$

An argument similar to that of [10, p. 166] shows that

$$(3.5) \quad (T - z)^{-1}E(\alpha)x \text{ is (weakly) continuous in } [\rho(T) \cup \{z: \text{Re}(z) \text{ outside } \alpha^-\}] - \Pi.$$

4. Proof of Theorem 2. Let $Z = \sigma(T) \cap \partial A$, so that, by (1.4), Z is a closed subset of ∂A of (arc length) measure 0. Let L denote the linear manifold of vectors x for which $\|(T - z)^{-1}x\|$ is bounded and $(T - z)^{-1}x$ is weakly continuous on $\partial A - Z$. Then one may define the "projection"

$$(4.1) \quad Px = -(2\pi i)^{-1} \int_{\partial A} (T - z)^{-1}x dz, \quad x \in L,$$

where, in fact, (Px, y) exists as a Riemann integral for x in L and y in \mathfrak{H} . Further, it is clear that L is invariant under T . It will first be shown that if $z_0 \in \sigma(T) \cap A$ then there exist vectors $x_n \in L$ ($n = 1, 2, \dots$) satisfying

$$(4.2) \quad (T - z_0)Px_n \rightarrow 0, \quad \|Px_n\| = 1 \quad (x_n \in L).$$

To see this, note that if $z_0 \in \sigma(T) \cap A$ is in the point spectrum of T then $(T - z_0)x = 0$ and hence $(T - z)^{-1}x = (z_0 - z)^{-1}x$ for some unit vector x and for $z \in \rho(T)$. Clearly, $x \in L$ and $Px = x$, so that (4.2) holds with all $x_n = x$. Further, if there exists a sequence $\{z_n\}$ such that $z_n \rightarrow z_0$, $z_n \in \sigma(T) \cap A$, and each z_n belongs to the point spectrum of T , then $(T - z_n)u_n = 0$, where $\|u_n\| = 1$. Since $(T - z)^{-1}u_n = (z_n - z)^{-1}u_n$ for $z \in \rho(T)$, and since $z_n \in A$, then $u_n \in L$ and $(T - z_n)Pu_n = 0$ with $Pu_n = u_n$. Clearly, (4.2) holds with $x_n = u_n$.

Suppose then that $z_0 \in \sigma(T) \cap A$ and that some neighborhood of z_0 is free of the point spectrum of T . It can be supposed that T^* is completely hyponormal. (In fact, if T^* [hence T] has a normal part, this part can be treated separately. As noted earlier, if N is normal and if D is any open set such that $\sigma(N) \cap D \neq \emptyset$, then N has a reducing space on which its spectrum is $(\sigma(N) \cap D)^-$.) In view of (1.4), it is easy to see that for almost all real t the line $\{w: \text{Re}(w) = t\}$ does not intersect the set Z . Further, since T^* is completely hyponormal, then, as noted above, any open disk centered at z_0 must intersect $\sigma(T)$ in a set of positive planar measure. It follows that there exist $\{z_n\}$ ($n = 1, 2, \dots$) for which $z_n \rightarrow z_0$, $z_n \in \sigma(T) \cap A$ but no z_n is in the point spectrum

of T , and for which $\{w: \operatorname{Re}(w) = \operatorname{Re}(z_n)\} \cap Z = \emptyset$. If n is fixed, it is clear that there exists an open interval α (depending on n) which contains $\operatorname{Re}(z_n)$ and satisfies

$$(4.3) \quad \{w: \operatorname{Re}(w) \in \alpha^-\} \cap Z = \emptyset.$$

It follows from (3.4) and (3.5) that

$$(4.4) \quad \begin{aligned} &\text{if } x = E(\alpha)x, \text{ then } (T - z)^{-1}x \text{ is weakly continuous} \\ &\text{and } \|(T - z)^{-1}x\| \leq (\text{const})\|x\| \text{ for } z \in \partial A - Z, \end{aligned}$$

where "const" depends on n but is independent of x or of α , provided that the length of α is sufficiently small. In particular, $x = E(\alpha)x \in L$, so that Px of (4.1) is defined.

Since T^* is hyponormal, then $T_z T_z^* - T_z^* T_z \geq 0$, where $T_z = T - z$ and z is arbitrary. Since $z_n \in \sigma(T)$ and z_n is not in the point spectrum of T , it follows that, for each fixed n , there exists a sequence of unit vectors $\{u_k\}$ for which

$$(4.5) \quad \text{both } (T^* - \bar{z}_n)u_k \rightarrow 0 \text{ and } (T - z_n)u_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

If $z = z_n$ in (3.3), it is seen that $(H - t_n)u_k \rightarrow 0$ ($t_n = \operatorname{Re}(z_n)$) and consequently, for any fixed (sufficiently small) open interval α containing t_n , $u_k - E(\alpha)u_k \rightarrow 0$. Hence, it can be assumed that (4.5) holds with

$$(4.6) \quad u_k = E(\alpha_k)u_k, \quad \|u_k\| = 1 \quad (k = 1, 2, \dots), u_k \in L,$$

where α_k is an open subinterval of α containing t_n , relation (4.3) holds with α replaced by α_k , and where $|\alpha_k| \rightarrow 0$ as $k \rightarrow \infty$.

Now, $(T - z)u_k = y_k + (z_n - z)u_k$, where $y_k = (T - z_n)u_k \rightarrow 0$ as $k \rightarrow \infty$ (n fixed) and hence, since $z_n \in A$,

$$(4.7) \quad \begin{aligned} u_k &= (2\pi i)^{-1} \int_{\partial A} (z - z_n)^{-1} u_k dz \\ &= Pu_k + (2\pi i)^{-1} \int_{\partial A} (z - z_n)^{-1} (T - z)^{-1} y_k dz. \end{aligned}$$

For $z \in \partial A - Z$, $(T - z)^{-1}y_k = u_k + (z - z_n)(T - z)^{-1}u_k$, so that

$$\|(T - z)^{-1}y_k\| \leq \|u_k\| + |z - z_n| \|(T - z)^{-1}u_k\|$$

and hence, by (4.4) with $x = u_k$ of (4.6),

$$(4.8) \quad \|(z - z_n)^{-1}(T - z)^{-1}y_k\| \leq \text{const} \quad \text{for } z \in \partial A - Z,$$

where "const" is independent of k (for $|\alpha_k|$ sufficiently small). Since $y_k \rightarrow 0$ as $k \rightarrow \infty$,

$$(4.9) \quad \|(z - z_n)^{-1}(T - z)^{-1}y_k\| \rightarrow 0 \text{ as } k \rightarrow \infty \quad \text{for } z \in \partial A - Z.$$

By (4.7),

$$\|u_k - Pu_k\| \leq (2\pi)^{-1} \int_{\partial A} \|(z - z_n)^{-1}(T - z)^{-1}y_k\| |dz|.$$

Hence, by (4.8), (4.9) and Lebesgue's uniformly bounded convergence theorem, $u_k - Pu_k \rightarrow 0$ as $k \rightarrow \infty$. Thus, by (4.5), one can suppose that $(T - z_n)Pu_k \rightarrow 0$ as $k \rightarrow \infty$ (n fixed), where $u_k = u_k^{(n)} \in L$ and $\|Pu_k\| = 1$. Since $z_n \rightarrow z_0$ as $n \rightarrow \infty$, it is clear that there exists a subsequence $\{x_n\}$ of $\{u_k^{(n)}\}$ satisfying (4.2).

Next, let \mathfrak{M} denote the closure of the linear manifold of vectors $\{Px\}$ where $x \in L$. Clearly, \mathfrak{M} is a hyperinvariant subspace of T . Since z_0 is an arbitrary point of $\sigma(T) \cap A$, it follows from (4.2) that $E \subset \sigma(T|_{\mathfrak{M}})$. Since \mathfrak{M} is hyperinvariant for T then, in particular, \mathfrak{M} is invariant under $(T - z)^{-1}$ whenever $z \in \rho(T)$, and hence $\sigma(T|_{\mathfrak{M}}) \subset \sigma(T)$. The proof of (3.1) and hence of Theorem 2 will be completed by showing that

$$(4.10) \quad \sigma(T) - E \subset \rho(T|_{\mathfrak{M}}).$$

In order to prove (4.10), let $z_1 \in \sigma(T) - E$, so that $z_1 \in \sigma$ where σ is some isolated part of $\sigma(T)$ contained in $\mathbf{C} - A^-$. Let D denote an associated admissible domain satisfying (1.1) and such that ∂D lies in $\mathbf{C} - A^-$. Next, let $x \in L$ and consider the vector function

$$F(z) = -(2\pi i)^{-1} \int_{\partial A} (t - z)^{-1}(T - t)^{-1}x \, dt.$$

Clearly, $F(z)$ is analytic in $\mathbf{C} - A^-$ and hence, by the Cauchy integral formula,

$$F(z_1) = (2\pi i)^{-1} \int_{\partial D} (z - z_1)^{-1}F(z) \, dz.$$

Also, $(T - z)F(z) = Px$ for $z \in \mathbf{C} - A^-$. If Q is defined by

$$Q = (2\pi i)^{-1} \int_{\partial D} (z - z_1)^{-1}(T - z)^{-1} \, dz,$$

then Q is clearly a bounded operator on \mathfrak{H} commuting with T and satisfying $QP_x = F(z_1)$. Hence, $Q(T - z_1)P_x = (T - z_1)QP_x = P_x$, and so, $Q(T - z_1)y = (T - z_1)Qy = y$ for any y in \mathfrak{M} . Thus $z_1 \in \rho(T|_{\mathfrak{M}})$, that is, (4.10). This completes the proof of Theorem 2.

5. An example. It will be shown that there exists an operator T with spectrum given by (1.5) and with the property that (1.6) holds for every invariant subspace of T . First, consider the sequence of nilpotent matrices

$$A_1 = 0, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \dots,$$

where each A_n is regarded as an operator on the corresponding n -dimensional Hilbert space \mathfrak{H}_n . Thus, $\sigma(A_n) = 0$ for $n = 1, 2, \dots$. Let $D = \{z: |z| < 1\}$ and, for each n , choose a polynomial $p_n(z)$ such that

$$(5.1) \quad p_n(0) = -1 + 1/n, \quad p_n(D) \subset \{z: |z| \leq 1 + 1/n\} \text{ and, for each } z \text{ in } D, \\ \text{there exists a } w (= w(z)) \text{ in } D \text{ satisfying } |z - p_n(w)| < 1/n.$$

That this can be done can be seen as follows. First, note that D can be mapped conformally onto itself by a function $f_n(z)$ for which $f_n(0) = -1 + 1/n$ and such that the mapping extends to a homeomorphism of D^- onto itself. (In fact, $f_n(z)$ can even be chosen to be a linear fractional transformation.) By Mergelyan's theorem (or less), the extended function $f_n(z)$ can be uniformly approximated on D^- by polynomials, and hence there exists a polynomial $q_n(z)$ for which $|f_n(z) - q_n(z)| < 1/2n$ for z in D^- ; cf. Rudin [15, p. 386]. If $c_n = -1 + 1/n - q_n(0)$, then $|c_n| < 1/2n$ and so $p_n(z) = q_n(z) + c_n$ satisfies (5.1).

Next, note that $\|A_n\| = 1$ for all n and hence, by (5.1) and the von Neumann-Heinz theorem (cf. Halmos [6, p. 123]),

$$(5.2) \quad \|p_n(A_k)\| \leq 1 + 1/n \quad (n, k \text{ arbitrary positive integers}),$$

where $p_n(A_k)$ is regarded as an operator on \mathfrak{H}_k . If

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

then A is the adjoint of the unilateral shift, $\sigma(A) = D^-$ and hence $\sigma(p_n(A)) = p_n(\sigma(A)) = p_n(D^-)$; further, every point of D is in the point spectrum of A .

Next, let $\{z_1, z_2, \dots\}$ be a dense subset of D . Clearly, for each fixed $n = 1, 2, \dots$, one can choose a positive integer k_n , together with n unit vectors $x_{nj} \in \mathfrak{H}_{k_n}$ ($j = 1, \dots, n$) such that $k_1 < k_2 < \dots$ and

$$(5.3) \quad \|(p_n(A_{k_n}) - z_j)x_{nj}\| < 2/n, \quad j = 1, \dots, n.$$

Let $T = \sum_{n=1}^{\infty} \bigoplus B_n$ on $\mathfrak{H} = \sum_{n=1}^{\infty} \bigoplus \mathfrak{H}_{k_n}$, where $B_n = (-1 + 2/n)I_{k_n} - p_n(A_{k_n})$ on \mathfrak{H}_{k_n} . Since $\sigma(A_k) = \{0\}$ for all k , it is clear from (5.1) that $\sigma(B_n) = \{-1 + 2/n + 1 - 1/n\} = \{1/n\}$. In view of (5.2), $\limsup_{n \rightarrow \infty} \|B_n + I_{k_n}\| \leq 1$ and hence $\sigma(T) \subset \{z: |z + 1| \leq 1\} \cup \{1, 1/2, 1/3, \dots\}$. This fact combined with (5.3) now yields (1.5).

Finally, it will be shown that if $\mathfrak{M} (\neq 0)$ is an invariant subspace of T then (1.6) holds. Let P_j ($j = 1, 2, \dots$) denote the projection of $\mathfrak{H} = \sum \bigoplus \mathfrak{H}_{k_n}$

onto \mathfrak{H}_{k_j} . Clearly, $P_j T = T P_j$ and $\mathfrak{M}_j = P_j \mathfrak{M} (\subset \mathfrak{H}_{k_j})$ is an invariant subspace of T (and B_j). Since $\mathfrak{M} \neq 0$, then $\mathfrak{M}_p \neq 0$ for some p . Also $T|_{\mathfrak{M}_p} = B_p|_{\mathfrak{M}_p}$ and, since $\partial\sigma(B_p|_{\mathfrak{M}_p}) \subset \sigma(B_p|_{\mathfrak{H}_{k_p}}) = \{1/p\}$, it follows that $\sigma(B_p|_{\mathfrak{M}_p}) = \{1/p\}$ and hence $\sigma((T+I)|_{\mathfrak{M}_p}) = \{1 + 1/p\}$. Consequently, if $S = T + I$,

$$1 + 1/p = \lim_{n \rightarrow \infty} \|(S|_{\mathfrak{M}_p})^n\|^{1/n} = \lim_{n \rightarrow \infty} \|(P_p S|_{\mathfrak{M}})^n\|^{1/n} \leq \lim_{n \rightarrow \infty} \|(S|_{\mathfrak{M}})^n\|^{1/n},$$

and so the spectral radius of $S|_{\mathfrak{M}}$ is not less than $1 + 1/p$. This means that $\sigma(T|_{\mathfrak{M}})$ is not a subset of $\{z: |z + 1| \leq 1\}$, and, since $\partial\sigma(T|_{\mathfrak{M}}) \subset \sigma(T)$, relation (1.6) follows.

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